Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 3, 1997

CORRECTNESS OF AN ASYMPTOTIC SOLUTION FOR LAMINAR FLOW OF A FLUID BETWEEN TWO ROTATING DISKS

E. V. Semenov

UDC 66.067.5

The present paper deals with an asymptotic solution as a power series in a nondimensional radial coordinate for the divergent flow of a viscous incompressible fluid which is induced by a linear source located at the axis of rotation of two coaxial disks. Convergence of the solution is estimated.

The characteristic properties of the flow of a condensed fluid in the space between two rotating equidistant disks or cones are of great importance in applied mechanics for analysis of the kinetics of these flows in various centrifugal pressure devices, separators, centrifuges, etc.

The development of fluid flow in the clearance between two rotating equidistant disks or cones has been investigated in [1–6]. Note that, as is often done for divergent flows, in the analysis of the causes of flow by a point or axial source the expansion of the desired solution into an inverse power series of a nondimensional radial coordinate is often used as an effective quantitative method. For an axially symmetric flow regime, this method offers the possibility of reducing the original problem to a one-dimensional one in the transverse coordinate. In this case, the solution is a parametric function of the radial coordinate. This makes it possible to reduce a boundary-value problem, for example, in an unbounded domain for the Navier–Stokes equations, to an infinite system of linear inhomogeneous ordinary differential equations.

The problem of fluid flow between two unbounded disks (planes) rotating with the same or different angular speeds can be conveniently considered in the cylindrical coordinate system $r\vartheta z$ fixed to one of the disks. In what follows, we assume without loss of generality that the disks and the fluid are suddenly driven to rotation with a constant angular velocity ω (Fig. 1).

Let u, v, and w be the radial, relative circumferential, and transverse components of the fluid velocity V, p is the pressure, ρ is the density, and ν is the coefficient of kinematic viscosity. If the origin of coordinates is located at the midpoint between the two disks at the axis of rotation, then the following non-slip boundary conditions can be used for the fluid at the disk walls:

$$u = v = w = 0 \quad \text{for} \quad z = \pm h/2. \tag{1}$$

Here h is the distance between the disks. In addition, the condition

$$V = 0 \quad \text{for} \quad r \to \infty \tag{2}$$

must be satisfied.

From here on, we use nondimensional quantities, choosing $l = \sqrt{\nu/\omega}$ as the characteristic length and $U = \sqrt{\nu\omega}$ as the characteristic velocity [5]. In this case, $r = l\bar{r}$, $z = lx$, $u = U\bar{u}$, $v = U\bar{v}$, $w = U\bar{w}$, $\Phi = U^2\bar{\Phi}$, and $\Phi = p/\rho - \omega^2 r^2/2$.

Below, if we drop all bars, then the equations of motion and continuity [7] for a steady axially symmetric flow of a viscous incompressible fluid in nondimensional form are written as

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial x} - \frac{v^2}{r} = -\frac{\partial \Phi}{\partial r} + 2v + \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial r}\Big[\frac{1}{r}\,\frac{\partial (ru)}{\partial r}\Big],$$

Moscow State Academy of the Food Industry, Moscow 125080. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 3, pp. 55–63, May–June, 1997. Original article submitted August 17, 1995; revision submitted January 5, 1996.

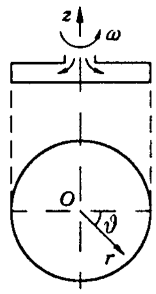

Fig. 1

$$\frac{u}{r}\frac{\partial}{\partial r}(rv)+w\frac{\partial v}{\partial x}=-2u+\frac{\partial^2 v}{\partial x^2}+\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right], \tag{3}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial x}=-\frac{\partial\Phi}{\partial x}+\frac{\partial^2 w}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \quad \frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial x}=0.$$

Since the flow-velocity field is symmetric about the plane $x = 0$, we can replace the boundary conditions (1) by

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=0, \quad w=0 \quad \text{for} \quad x=0; \tag{4}$$

$$u=v=w=0 \quad \text{for} \quad x=\mu, \quad \mu=\lambda/2, \quad \lambda=h\sqrt{\omega/\nu}. \tag{5}$$

Taking into account (2), we seek the functions u and v as the series

$$u(r,x)=u_1(x)/r+u_2(x)/r^2+\ldots, \quad v(r,x)=v_1(x)/r+v_2(x)/r^2+\ldots \; . \tag{6}$$

In turn, we have

$$\Phi(r,x)=\Phi_1(x)\ln r+\Phi_2(x)/r+\ldots \; . \tag{7}$$

In accordance with (4)–(6), we have the following boundary conditions for u_n and v_n ($n = 1, 2, \ldots$):

$$\frac{\partial u_n}{\partial x}=\frac{\partial v_n}{\partial x}=0 \quad \text{for} \quad x=0; \tag{8}$$

$$u_n=v_n=0 \quad \text{for} \quad x=\mu. \tag{9}$$

We write the condition of fluid rate Q in nondimensional form

$$q=2\pi\int_0^\mu ur\,dx, \quad q=0.5QU^{-1}l^{-2}. \tag{10}$$

Taking into account (6), by virtue of (10) we obtain for u_n

$$\int_0^\mu u_1(x)\,dx=q/(2\pi); \tag{11}$$

$$\int_0^\mu u_n(x)\,dx=0, \quad n=2,3,\ldots \; . \tag{12}$$

In integrating the last equation of system (3) over x, we obtain

$$w=-\frac{1}{r}\frac{\partial}{\partial r}\int_0^x ur\,dx.$$

In accordance with (10), the boundary conditions (4) and (5) for w are satisfied. In this case, taking into account (6), we have

$$w(r,x)=\frac{1}{r^3}\int_0^x u_2(x)\,dx+\frac{2}{r^4}\int_0^x u_3(x)\,dx+\ldots\,. \tag{13}$$

Substituting expansions (6), (7), and (13) into the first three equations of system (3), we write

$$-\Big(\frac{u_1}{r}+\frac{u_2}{r^2}+\ldots\Big)\Big(\frac{u_1}{r^2}+\frac{2u_2}{r^3}+\ldots\Big)+\int_\mu^x\Big(\frac{u_2}{r^3}+\frac{2u_3}{r^4}+\ldots\Big)\,dx\Big(\frac{u_1'}{r}+\frac{u_2'}{r^2}+\ldots\Big)-\frac{1}{r}\Big(\frac{v_1}{r}+\frac{v_2}{r^2}+\ldots\Big)$$

$$=-\frac{\Phi_1}{r}+\frac{\Phi_2}{r^2}+\ldots+2\Big(\frac{v_1}{r}+\frac{v_2}{r^2}+\ldots\Big)+\frac{u_1''}{r}+\ldots+\frac{1\cdot 3}{r^4}u_2+\frac{2\cdot 4}{r^5}u_3+\ldots; \tag{14}$$

$$-\Big(\frac{u_1}{r}+\frac{u_2}{r^2}+\ldots\Big)\Big(\frac{v_2}{r^2}+\frac{2v_3}{r^3}+\ldots\Big)+\int_\mu^x\Big(\frac{u_2}{r^3}+\frac{2u_3}{r^4}+\ldots\Big)\,dx\Big(\frac{v_1'}{r}+\ldots\Big)$$

$$=-2\Big(\frac{u_1}{r}+\frac{u_2}{r^2}+\ldots\Big)+\frac{v_1''}{r}+\frac{v_2''}{r^2}+\ldots+\frac{1\cdot 3}{r^4}v_2+\frac{2\cdot 4}{r^5}v_3+\ldots; \tag{15}$$

$$-\Big(\frac{u_1}{r}+\frac{u_2}{r^2}+\ldots\Big)\int_\mu^x\Big(\frac{1\cdot 3}{r^4}u_2+\frac{2\cdot 4}{r^5}u_3+\ldots\Big)\,dx+\int_0^x\Big(\frac{u_2}{r^3}+\frac{2u_3}{r^4}+\ldots\Big)\,dx\Big(\frac{u_2}{r^3}+\frac{2u_3}{r^4}+\ldots\Big)$$

$$=-\Phi_1'\ln r-\Phi_2'/r-\ldots+\frac{u_2'}{r^3}+\frac{2u_3'}{r^4}+\ldots+\int_0^x\Big(\frac{1\cdot 3^2}{r^5}u_2+\frac{2\cdot 4^2}{r^6}u_3+\ldots\Big)\,dx. \tag{16}$$

Here primes denote derivatives with respect to x. We group terms with the same r-dependent factors and obtain, from (14)–(16), the following system:

$$\Phi_1=C_1/2=\text{const}; \tag{17}$$

$$u_1''+2v_1=\Phi_1,\quad v_1''-2u_1=0,\quad \Phi_2'=0,\quad \Phi_2=-C_2/2; \tag{18}$$

$$u_2''+2v_2=-\Phi_2,\quad v_2''-2u_2=0,\quad \Phi_3=-C_3/4; \tag{19}$$

$$u_3''+2v_3=-2\Phi_3-u_1^2-v_1^2,\quad v_3''-2u_3=0,\quad -\Phi_4'+u_2'=0,\quad \Phi_4=u_2-C_4/6; \tag{20}$$

$$\begin{gathered} u_4''+2v_4=-3\Phi_4-u_1u_2-2u_1u_2-2v_1v_2+1\cdot 3u_2,\\ v_4''-2u_4=-u_1v_2-v_1'\int_0^x u_2\,dx-3v_2,\quad -\Phi_5'+2u_3'=0; \end{gathered} \tag{21}$$

$$\begin{gathered} u_5''+2v_5=-4\Phi_5-3u_1u_3-2u_2^2-u_3u_1+u_2'\int_0^x u_2\,dx+2u_1'\int_0^x u_3\,dx-2v_1v_3-v_2^2+2\cdot 4u_3,\\ v_5''-2u_5=-2u_1v_3-u_2v_2+v_2'\int_0^x u_2\,dx+2v_1'\int_0^x u_3\,dx-2\cdot 4v_3,\quad -\Phi_6'+3u_4+1\cdot 3^2\int_0^x u_2\,dx=0; \end{gathered} \tag{22}$$

$$\begin{gathered} u_6''+2v_6=-5\Phi_6-4u_1u_4-3u_2u_3-2u_3u_2-u_4u_1+3u_1'\int_\mu^x u_4dx+2u_2'\int_\mu^x u_3\,dx+u_3'\int_\mu^x u_2dx-2v_1v_4-2v_2v_3+3\cdot 5u_4,\\ v_6''-2u_6=-3u_1v_4-2u_2v_3-u_3v_2+v_3'\int_\mu^x u_2\,dx+2v_2'\int_\mu^x u_3\,dx+3v_1'\int_\mu^x u_4\,dx-3\cdot 5v_3, \end{gathered} \tag{23}$$

$$-2 \cdot 4u_1 \int_0^x u_3\,dx - 1 \cdot 3u_2 \int_0^x u_2\,dx + u_2 \int_0^x u_2\,dx = -\Phi_7' + 4u_5' + 2 \cdot 4^2 \int_0^x u_3\,dx;$$

$$\ldots .$$

Since it follows from system (17)–(23) and boundary conditions (3) and (9) that even approximations in the velocities u_{2m} and v_{2m} are identically zero, the system for odd approximations is of the form

$$u_n'' + 2v_n = -(n-1)\Phi_n - \sum_i^{n-2} iu_iu_{n-1-i} + \sum_i^{n-3} iu_{n-2-i}' \int_\mu^x u_{i+1}\,dx - \sum_i^{n-1} v_iv_{n-1-i} + (n-3)(n-1)u_{n-2}; \quad (24)$$

$$v_n'' - 2u_n = -\sum_i^{n-3} iu_{n-2-i}v_{i+1} + \sum_i^{n-3} iv_{n-2-i}' \int_\mu^x u_{i+1}\,dx - (n-3)(n-1)v_{n-2}; \quad (25)$$

$$\sum_i^{n-4} i(i+2)u_{n-3-i} \int_\mu^x u_{i+1}\,dx + \sum_i^{n-5} (n-4-i)u_{n-3-i} \int_\mu^x iu_{i+1}\,dx$$

$$= -\Phi_{n+1}' + (n-2)u_{n-1}' + (n-4)(n-2)^2 \int_\mu^x u_{n-3}\,dx. \quad (26)$$

Here $n = 2m+1$, $i = 2k+1$, where k and $m = 0, 1, 2, \ldots$. Summation is performed over positive subscripts. Following from (25) and (26), we obtain

$$u_n = \frac{1}{2}\left[v_n'' + \sum_{i=1}^{n-3} i\left(u_{n-2-i}v_{i+1} - v_{n-2-i}' \int_\mu^x u_{i+1}\,dx\right) + (n-3)(n-1)v_{n-2}\right]; \quad (27)$$

$$\Phi_n = -\frac{C_n}{2(n-1)} + (n-3)u_{n-2} + (n-5)(n-3)^2 \int_0^x\int_\mu^x u_{n-4}\,d\xi\,dx$$

$$- \sum_i^{n-5} i(i+2) \int_0^x u_{n-4-i} \int_\mu^x u_{i+1}\,d\xi\,dx - \sum_i^{n-6} (n-5-i) \int_0^x u_{n-4-i} \int_\mu^x iu_{i+1}\,d\xi\,dx. \quad (28)$$

Thus, by virtue of (27) and (28), instead of (24) we have

$$v_n^{\mathrm{IV}} + 4v_n = C_n + f_n(x), \quad (29)$$

where C_n is a constant and $f_1 = 0$;

$$f_n(x) = -2\sum_i^{n-2} (v_iv_{n-1-i} + iu_iu_{n-1-i}) + \sum_i^{n-3} i\left[-(u_{n-2-i}v_{i+1})'' + \left(v_{n-2-i}' \int_\mu^x u_{i+1}\,dx\right)''\right.$$

$$\left. + 2u_{n-2-i}' \int_\mu^x u_{i+1}\,dx\right] + 2(n-1)\sum_i^{n-6} (n-5-i) \int_0^x u_{n-4-i} \int_\mu^x iu_{i+1}\,d\xi\,dx$$

$$-2(n-1)\sum_i^{n-5} i(i+2) \int_0^x u_{n-4-i} \int_\mu^x u_{i+1}\,d\xi\,dx - (n-1)(n-3)v_{n-2}'' - 2(n-1)(n-3)(n-5) \int_0^x\int_\mu^x u_{n-4}\,d\xi\,dx. \quad (30)$$

From (8), (9), and (27) and based on the boundary conditions for v_n we obtain additionally

$$v_n'''(0) = 0, \qquad v_n''(\mu) = 0. \quad (31)$$

In addition, according to (12) and (27) we have

$$v_n'(\mu) = v_{n**}'. \quad (32)$$

Here

$$v'_{n**}(\mu) = \sum_{i}^{n-3} i \left(-\int_0^{\mu} u_{n-2-i} v_{i+1}\, dx + \int_0^{\mu} v'_{n-2-i} \int_{\mu}^{x} u_{i+1}\, d\xi\, dx \right) - (n-3)(n-1) \int_0^{\mu} v_{n-2}\, dx. \tag{33}$$

We write the solution of Eq. (29) which is subject to the boundary conditions (8), (9), (31), and (32) in the following form [8]:

$$v_n(x) = C_{n1} y_1(x) + \ldots + C_{n4} y_4(x) + \int_0^{x} y_4(x-\xi)[C_n + f_n(\xi)]\, d\xi. \tag{34}$$

Here

$$y_1 = \cosh x \cos x, \quad y_2 = (\cosh x \sin x + \sinh x \cos x)/2,$$
$$y_3 = \sinh x \sin x/2, \quad y_4 = (\cosh x \sin x - \sinh x \cos x)/4. \tag{35}$$

In this case, $C_{n2} = C_{n4} = 0$ according to (8) and (31). Instead of (34), we have

$$v_n = \bar{v}_n + v_{n*}. \tag{36}$$

Here

$$\bar{v}_n = C_{n1} y_1(x) + C_{n3} y_3(x) + C_n[1 - y_1(x)]/4, \quad v_{n*} = \int_0^{x} y_4(x-\xi) f_n(\xi)\, d\xi. \tag{37}$$

Note that, according to (9), (31), and (32), we obtain the following system to determine the coefficients C_{n1}, C_{n3}, and C_n:

$$C_{n1}y_1 + C_{n3}y_3 + C_n(1-y_1)/4 = J_4, \quad -4C_{n1}y_3 + C_{n3}y_1 + C_n y_3 = J_2, \quad -4C_{n1}y_4 + C_{n3}y_2 + C_n y_4 = J_3. \tag{38}$$

Here

$$J_4 = -v_{n*}, \quad J_2 = -v''_{n*}, \quad J_3 = -v'_{n*} + v'_{n**}; \tag{39}$$

$$v_{n*}^{(k)} = \int_0^{\mu} y_k(x-\xi) f_n(\xi)\, d\xi, \quad y_k = y_k(\mu), \quad k = 1, 2, 3, 4. \tag{40}$$

The solution of system (38) is as follows:

$$C_{n1} = \Delta_{n1}/\Delta, \quad C_{n3} = \Delta_{n3}/\Delta, \quad C_n = \Delta_n/\Delta. \tag{41}$$

Here

$$\Delta = y_1 y_4 - y_2 y_3 = (\sin 2\mu - \sinh 2\mu)/8; \tag{42}$$

$$\Delta_{n1} = (y_1y_4 - y_2y_3)J_4 + [y_2(1-y_1)/4 - y_3y_4]J_2 + [y_3^2 - y_1(1-y_1)/4]J_3,$$
$$\Delta_{n3} = y_4J_2 - y_3J_3, \quad \Delta_n = 4(y_1y_4 - y_2y_3)J_4 - (y_1y_2 + 4y_3y_4)J_2 + (y_1^2 + 4y_3^2)J_3. \tag{43}$$

In accordance with (11), (37), (40), and (41), for the first approximation we have

$$v_1(x) = A_1 y_1(x) + B_1 y_3(x) + C_1/4; \tag{44}$$

$$u_1(x) = v_1''(x)/2 = -4A_1 y_3(x) + B_1 y_1(x), \quad \Phi_1 = C_1/2, \tag{45}$$

where

$$A_1 = C_{11} - C_1/4, \quad B_1 = C_{13}, \quad C_{11} = q(\sinh^2\mu + \cos^2\mu - \cosh\mu\cos\mu)/(8\pi\Delta),$$
$$C_{13} = -q\sinh\mu\sin\mu/(4\pi\Delta), \quad C_1 = q(\cos^2\mu + \sinh^2\mu)/(2\pi\Delta). \tag{46}$$

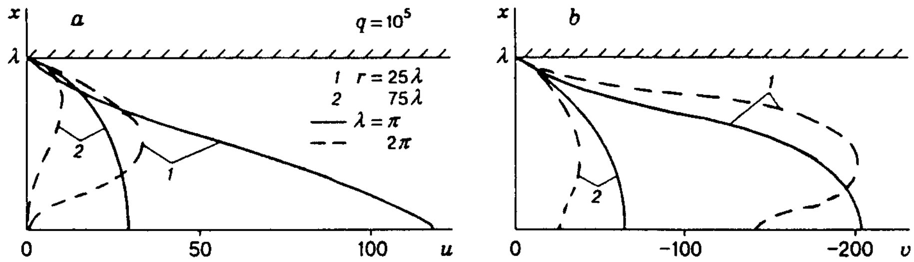

Fig. 2

By virtue of (33), $v'_{3**} = 0$, and, by virtue of (40),

$$v_{3*} = -\frac{D_3}{20}(\cosh 2x + \cos 2x - 2y_1) - \frac{C_1}{16}[4A_1 x y_4 + B_1(2y_3 - xy_2)] + \frac{C_1^2}{32}(y_1 - 1),$$

$$v'_{3*} = -\frac{D_3}{10}(\sinh 2x - \sin 2x + 4y_4) - \frac{C_1}{16}[4A_1(y_4 + xy_3) + B_1(y_2 - xy_1)] - \frac{C_1^2}{8} y_4,$$

$$v''_{3*} = -\frac{D_3}{5}(\cosh 2x - \cos 2x + 4y_3) - \frac{C_1}{16}[4A_1(2y_3 + xy_2) + 4B_1 x y_4] - \frac{C_1^2}{8} y_3.$$

Here $y_k = y_k(x)$. According to (27) and (36), we therefore have

$$v_3 = \bar{v}_3 + v_{3*}, \quad u_3 = (\bar{v}''_3 + v''_{3*})/2, \quad \Phi_3 = -C_3/4, \tag{47}$$

where $\bar{v}_3 = C_{31}y_1 + C_{33}y_3 + C_3(1 - y_1)/4$, $C_{31} = \Delta_{31}/\Delta$, $C_{33} = \Delta_{33}/\Delta$, and $C_3 = \Delta_3/\Delta$; Δ_{31}, Δ_{33}, and Δ_3 are calculated using (43). According to (39), $J_4 = -v_{3*}(\mu)$, $J_2 = -v''_{3*}(\mu)$, and $J_3 = -v'_{3*}(\mu)$. In accordance with (13) and (27), for the transverse velocity component we obtain

$$w \approx \frac{2}{r^4}\int_0^x u_3(\xi)\,d\xi = \frac{v'_3(x)}{r^4}. \tag{48}$$

Thus, based on (6) and (7) and using the three approximations, we can take the following formulas as computational ones:

$$u \approx u_1(x)/r + u_3(x)/r^3, \quad v \approx v_1(x)/r + v_3(x)/r^3, \quad \Phi \approx \Phi_1 \ln r + \Phi_3/r^2. \tag{49}$$

Here u_1, u_3, v_1, v_3, Φ_1, and Φ_3 are found using formulas (44), (45), and (47).

Determination of the explicit fifth, seventh, and higher-order approximations is a rather difficult problem and is omitted here. Nevertheless, in principle, high-order approximations could be found explicitly in accordance with the iterative approximation scheme considered above with the use of modern computers and algorithmic languages of symbolic programming. Numerical simulation offers the possibility of investigating the convergence of the iterative solution obtained using successive approximation. Figure 2 shows calculation results by formulas (43) and (49) with the values of λ and q (Eckman and Rossby numbers, respectively), typical of the working hollows of plate separators or pressure disk devices.

Numerical calculations have shown the presence of the points of inflection in the profiles of the circumferential and radial velocity components for $\lambda > \pi$ and the appearance of reversed flows for $\lambda > 2\pi$, etc. [5]. A quantitative comparison has shown that the maximum difference between the first and third approximations at a distance of 50 clearances from the axis of rotation is approximately 5% for the circumferential velocity v and 30% for the radial velocity u.

The problem of convergence for expansions (6) and (7) is evidently of both practical and theoretical interest, because the proposed computational scheme realizes a solution to the full boundary-value problem (1)–(3). Let us estimate the convergence, for example, of the functional series for the circumferential velocity component (6) with r as a parameter. The largest absolute value of this component is reached in the middle of the clearance between the disks. We then take

$$v(0)=\sum_i^{\infty}\left|v_{2i-1}(0)\right|r^{1-2i}=\sum_i^{\infty}\left|C_{2i-1}\right|r^{1-2i} \tag{50}$$

as a majorant of the series. It is clear that if the convergence of expansion (50) is proved, then the convergence of the series (6) is also proved.

To simplify the problem, confining ourselves to small values of λ with accuracy up to the third order and using the Krylov functions (35) we have approximately

$$y_1=1-\frac{x^4}{6},\quad y_2=x-\frac{x^5}{30},\quad y_3=\frac{x^2}{2}-\frac{x^6}{180},\quad y_4=\frac{x^3}{6}-\frac{x^7}{1260}. \tag{51}$$

Therefore, from (44)-(46) we obtain in the first approximation

$$v_1=\varepsilon(x^2-\mu^2)(x^5-5\mu^2)/\Delta,\quad u_1=6\varepsilon(x^2-\mu^2)/\Delta, \tag{52}$$

where $\varepsilon=q/(48\pi)$ and $\Delta=-\mu^3/3$. In addition, instead of formulas (43), we write approximately

$$\begin{gathered}\Delta_{n1}=\mu^3(8v_{n*}+\mu^2v''_{n*}-5\mu v'_{n*})/24+5\mu^4v_{n**}/24,\\ \Delta_{n3}=\mu^3(3v'_{n*}-\mu v''_{n*})/6-\mu^2v_{n**}/2,\quad \Delta_n=(4\mu^3v_{n*}+3\mu v''_{n*}-3v'_{n*})/3+v_{n**}.\end{gathered} \tag{53}$$

From (41), (51), and (53) we have

$$\begin{gathered}C_{n1}=-\frac{\mu^3}{144\Delta}\int_0^{\mu}(\mu^2+2\mu\xi-8\xi^2)(\mu-\xi)f_n(\xi)\,d\xi+\frac{5\mu^4}{24}v_{n**},\\ C_{n3}=-\frac{\mu^2}{12\Delta}\int_0^{\mu}(-\mu+3\xi)(\mu-\xi)f_n(\xi)\,d\xi-\frac{\mu^2}{2}v_{n**},\ C_n=\frac{1}{2\Delta}\int_0^{\mu}(\mu+\xi)(\mu-\xi)f_n(\xi)\,d\xi+v_{n**}.\end{gathered} \tag{54}$$

Since, as follows from (30), the term $-2\sum_i^{n-2}(v_iv_{n-1-i}+iu_iu_{n-1-i})$ contributes mainly, in the number of terms, to the value of the function $f_n(x)$, to simplify subsequent analysis of convergence, we further assume that

$$f_n\approx-2\sum_i^{n-2}(v_iv_{n-1-i}+iu_iu_{n-1-i}). \tag{55}$$

In addition, recalling the above assumption of the nondimensional clearance λ, we can replace (55) by the following approximate formula:

$$f_n=-2(n-2)\sum_i^{(n-3)/2}u_iu_{n-1-i}.$$

In this case, from (52) we find up to the 4th order of smallness that $f_3=-2u_1^2=-72\varepsilon^2(\mu^2-x^2)^2/\Delta^2$. Using the formula

$$\int_0^{\mu}x^k(\mu-x)^m\,dx=\frac{k!m!}{(k+m+1)!}\mu^{k+m+1},$$

we then have, in accordance with (54),

$$\begin{gathered}C_{31}=\frac{3!\varepsilon^2\mu^{11}}{2\Delta^3}\left(\frac{1!}{4!}+3\,\frac{1!1!}{5!}-5\,\frac{2!}{6!}-15\,\frac{3!}{7!}-8\,\frac{4!}{8!}\right),\\ C_{33}=\frac{6\cdot3!\varepsilon^3\mu^9}{\Delta^3}\left(-\frac{1!}{4!}+\frac{1!1!}{5!}+5\frac{2!}{6!}+3\frac{3!}{7!}\right),\quad C_3=-\frac{36\cdot3!\varepsilon^2\mu^7}{\Delta^3}\left(3\frac{1!}{4!}+7\frac{1!1!}{5!}+5\frac{2!}{6!}+\frac{3!}{7!}\right).\end{gathered} \tag{56}$$

The structure of the coefficients C_{n1}, C_{n3}, and C_n is also similar to relations (56) for the coefficients C_{31}, C_{33}, and C_3 if the function $f_n(x)$ is defined by formula (55).

In addition,

$$v_{3*} = -\int\limits_0^x \frac{(x-\xi)^3}{6} \frac{72\,\varepsilon^2}{\Delta^2} (\mu^2 - \xi^2)^2 \, d\xi = -\frac{12 \cdot 3!\varepsilon^2}{\Delta^2} \left(\frac{\mu^4 x^4}{4!} - 2 \cdot 2! \, \frac{\mu^2 x^8}{6!} + \frac{4! x^8}{8!} \right).$$

Hence, $v_3 = C_{31} y_1 + C_{33} y_3 + C_3 (1 - y_1)/4 + v_{3*}$.

We thus take the approximate value of the majorant (50) in the form

$$v(0) = \sum_i^{\infty} \left[\frac{\mu^3}{144\Delta} \left| \int\limits_0^{\mu} (\mu^2 + 2\mu\xi - 8\xi^2)(\mu - \xi) f_{2i-1}(\xi) \, d\xi \right| + \frac{5\mu^4}{24} \left| v_{2i-1,**} \right| \right] r^{1-2i}.$$

Although it follows from analysis of the expressions for the third (56) and subsequent approximations that the coefficients $C_{2m+1,1}$, $C_{2m+1,3}$, and C_{2m+1} at $m \to \infty$ form absolutely convergent numerical series, it is, however, difficult to estimate these series quantitatively. Evidently, the inequality $\lim\limits_{m \to \infty} |C_{2m+1}/C_{2m-1}| < r^2$ $(r \to \infty)$ is a consequence of their convergence. Therefore, the asymptotically uniform convergence of the series (6) is proved within the framework of the adopted assumptions.

The question whether the solution of boundary-value problem (1)–(3) constructed by the scheme (6) and (7) is unique remains open.

In conclusion, it should be mentioned that the kinetic characteristics of a steady divergent fluid flow in a hollow between two disks rotating with the same angular speed were determined experimentally by Adams and Rice et al. [9, 10] and Kohler [11]. For instance, the results of measurements of the pressure difference along the radius are given in [9, 10] for $\lambda = 2.25$ and $q \approx 10^6$, and the measurement data for the radial and circumferential velocity components are given in [11] for $\lambda = 4$ and $q \approx 10^5$. In addition, there is satisfactory agreement between the calculation results in Fig. 2 and the experiments, both quantitatively and qualitatively.

REFERENCES

1. W. Rice, "An analytical and experimental investigation of multiple disk pumps and compressors," *Trans. ASME. Ser. A.*, No. 3, 191–200 (1963).
2. J.-L. Peube, "Sur l'ecoulement radial permanent d'un fluide visqueux incompressible entre deux plans paralleles," *J. Méc.*, **2**, No. 4, 377–395 (1963).
3. J.-L. Peube and F. Kreith, L'ecoulement permanent d'un fluide visqueux incompressible entre deux disques paralleles en rotation," *J. Méc.*, **5**, 261–286 (1966).
4. F. Kreith, "Reverse transition in radial source flow between two parallel planes," *Phys. Fluids* **8**, 1189–1190 (1965).
5. E. V. Semenov, "Construction of an asymptotic solution for laminar fluid flow between the disks of a separator," in: *Theoretical Foundations of Chemical Technology*, No. 5 (1975), pp. 678–683.
6. R. Debuchy, A. Dument, and H. Muhe, "Radial inflow between a rotating and stationary disc," *Compets Rendus Acad. Sci. Ser. II*, No. 4, 437–442 (1993).
7. L. G. Loitsyanskii, *Fluid Mechanics* [in Russian], Nauka, Moscow (1970).
8. E. Von Kamke, *Differentialgleichungen Lösungsmethoden und Lsungen*, Leipzig (1959).
9. R. Adams and W. Rice, "An experimental investigation of flow between jointly rotating disks," *Proc. of Am. Soc. Eng.-Mech.*: Transl. J. *Trans. ASME. J. Ser. E*, No. 3, (1970), 272–277.
10. L. Pater, E. Grouther, and W. Rice, "Flow regime definition for flow between corotating disks" *Trans. ASME, Ser. I*, **196**, No. 1, 29–34 (1974).
11. M. Kohler, "Die Stromung durch des Spalteelement einer Reibungspumpe," *Stromungsmechanik und Stromungsmashinen*, No. 4, 24–25 (1971).